RECURRENCE RELATIONS

1 The Scottish Bank pays 9% compound interest p.a. on it's Super Saver account. A customer opens such an account with a deposit of £500. If B_n is the balance after n years:

(*a*) Write down a recurrence relation.

(*b*) Calculate B_1, B_2, B_3, B_4.

(*c*) Write down a formula for B_n.

(*d*) If no further deposits are made, after how many years will the value of the interest gained have exceeded the value of the initial deposit?

2 State whether the following recurrence relations have a limit and if so determine algebraically this limit.

(*a*) $u_0 = 8, \quad u_{n+1} = 0.25u_n + 4$ (*b*) $u_0 = 2, \quad u_{n+1} = 12 - 1.5u_n$

3 A recurrence relation is defined by $u_{n+1} = pu_n + q$ where $-1 < p < 1$ and $u_0 = 12$

(*a*) If $u_1 = 15$ and $u_2 = 16$ find the values of p and q

(*b*) Find the limit of this recurrence relation as $n \to \infty$

4 A hospital patient is put on medication which is taken once per day. The dose is 35mg and each day the patient's metabolism burns off 70% of the drug in her system. It is known that if the level of the drug in the patients system reaches 54mg then the consequences could be fatal. Is it safe for the patient to take the medication indefinitely?

5 The brake fluid reservoir in a car is leaky. Each day it loses 3% of its contents. To compensate for this daily loss the driver "tops up" once per week with 50ml of fluid. For safety reasons the level of fluid in the reservoir should always be between 200ml & 260ml. Initially the fluid level is 255ml.

(*a*) Find a recurrence relation to describe the above.

(*b*) Determine the fluid levels after 1 week and 4 weeks.

(*c*) Is the process effective in the long run?

1a) $B_{n+1} = 1.09 B_n$ 1

b) $B_0 = £500$
$B_1 = £545$
$B_2 = £594.05$
$B_3 = £647.51$
$B_4 = £705.79$ 1

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c) $B_n = 500 \times 1.09^n$ 1 1

d) $B_8 = £996.28$
$B_9 = £1085.95$ 1
ie. the interest after 9 years is £585.95 1
$\Rightarrow$ The interest exceeds the initial deposit after 9 years

2. a) $u_{n+1} = 0.25u_n + 4$
a limit exists as $-1 < 0.25 < 1$ 1
at the limit $L = 0.25L + 4$ 1 or $L = \frac{1}{4}L + 4$

$0.75L = 4$

$L = \frac{4}{0.75}$

$L = 5.3$ 1

$\frac{3}{4}L = 4$

$L = 4 . \frac{4}{3}$

$L = \frac{16}{3}$

b) $u_{n+1} = 12 - 1.5u_n$
a limit does not exist as $-1.5 < -1$ 1

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3. a) $u_0 = 12 \quad u_1 = 15 \quad u_2 = 16$

$u_{n+1} = pu_n + q$

$u_2 = pu_1 + q$
$16 = 15p + q$
$q = 16 - 15p$ 1

$u_1 = pu_0 + q$
$15 = 12p + q$
$q = 15 - 12p$ 1

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$\Rightarrow \quad 16 - 15p = 15 - 12p$
$1 = 3p$
$p = \frac{1}{3}$ 1 $\Rightarrow$ $q = 16 - 15\left(\frac{1}{3}\right)$
$q = 11$ 1

3b) $u_{n+1} = \frac{1}{3}u_n + 11$

a limit exists as $-1 < \frac{1}{3} < 1$

$\Rightarrow \quad L = \frac{1}{3}L + 11$

$\frac{2}{3}L = 11$

$L = 11 . \frac{3}{2}$

$L = \frac{33}{2}$

4 $u_{n+1} = 0.3u_n + 35$

a limit exists as $-1 < 0.3 < 1$

$\Rightarrow \quad L = 0.3L + 35$

$0.7L = 35$

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$L = \frac{35}{0.7}$

$L = 50$

It is safe to take the medicine indefinitely as the level of the drug in the body will not exceed 50mg.

5. a) Daily loss of 3% = 97% remaining each day

$(0.97)^7 = 0.81$

$\Rightarrow$ 81% remaining after one week

$\Rightarrow \quad u_{n+1} = 0.81 \; {}_n + 50$

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b) $u_1 = 257$ ml, $u_2 = 260$ ml

c) a limit exists as $-1 < 0.81 < 1$

$\Rightarrow \quad L = 0.81L + 50$

$0.19L = 50$

$L = \frac{50}{0.19}$

$L = 263$

In the long run the brake fluid will exceed the safety level as 263 ml > 260ml.